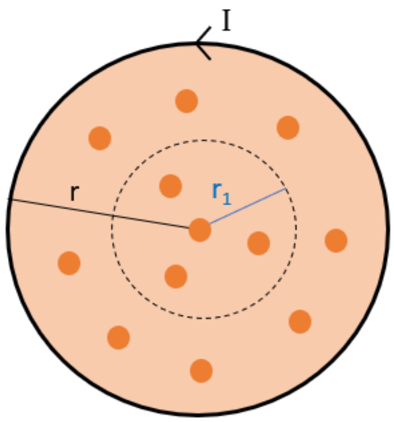
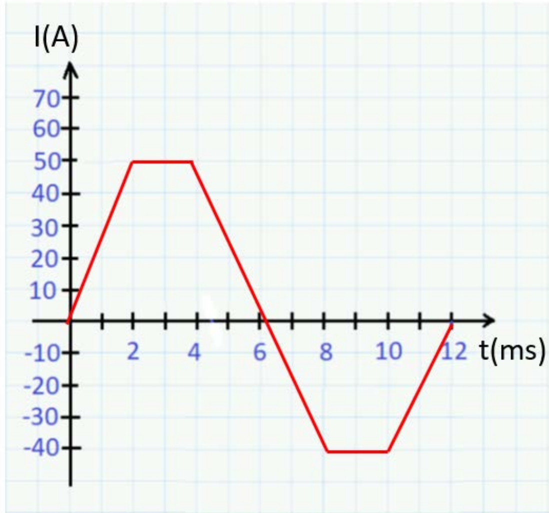
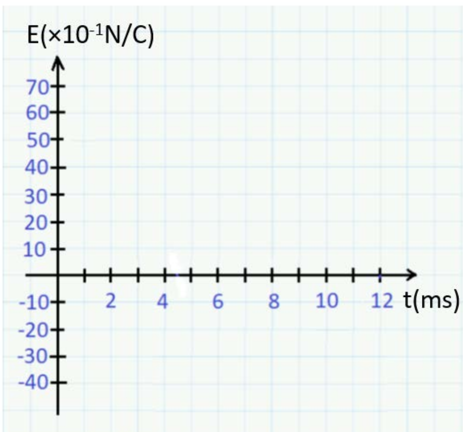
HW 12: Faraday's Law @ Maxwell-Ampere's Law

Problem 1. Consider the top-down view of a solenoid. It has an r = 10cm radius, and length ℓ = 20cm, 1000 turns, and a (presently) counter-clockwise current I, producing a (presently) out-of-the-page magnetic field B. To the right is plotted the current in the solenoid as a function of time (+ corresponds to CCW, and – to CW):





(a) Let r_1 = 5cm. Plot the electric field induced along this radius as a function of time (+ corresponds to CCW, and – to CW).



First we need to know what the field running through the solenoid is:

$$
\begin{aligned}
B &= \mu_0 n I \\
&= \left(4\pi \times 10^{-7}\,\frac{\mathrm{T\cdot m}}{\mathrm{A}}\right)\left(\frac{1000}{0.20\mathrm{m}}\right) I \\
&= 20\pi \times 10^{-4}\, I
\end{aligned}
$$

Then from Faraday's law,

$$\oint \mathbf{E}\cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}$$

$$\oint E dr = -\frac{d(\mathbf{B}\cdot\mathbf{A})}{dt}$$

$$E\cdot 2\pi r_1 = -\frac{dB}{dt}A$$

$$E = -\frac{A}{2\pi r_1}\frac{dB}{dt} = -\frac{\pi r_1^2}{2\pi r_1}\frac{dB}{dt} = -\frac{r_1}{2}\frac{dB}{dt}$$

$$= -\frac{r_1}{2}\cdot 20\pi\times10^{-4}\frac{dI}{dt} = -\frac{0.05}{2}\cdot 20\pi\times10^{-4}\frac{dI}{dt} = -1.57\times10^{-4}\frac{dI}{dt}$$

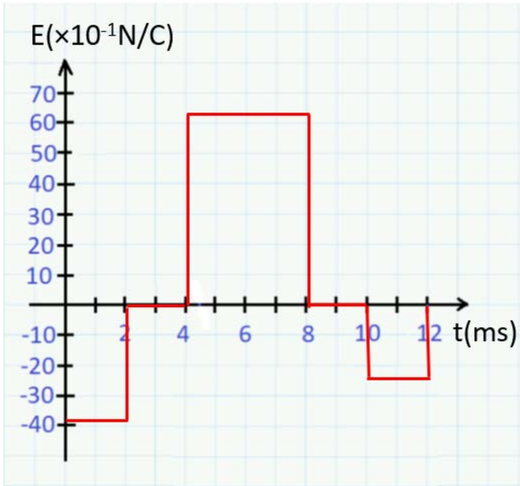
Of course dI/dt is just the slope of the I vs. t graph, which is

$$\frac{dI}{dt} = \begin{cases} 25\,\mathrm{kA/s} & (0\mathrm{s}, 2\mathrm{s}) \\ 0\,\mathrm{kA/s} & (2\mathrm{s}, 4\mathrm{s}) \\ -40\,\mathrm{kA/s} & (4\mathrm{s}, 8\mathrm{s}) \\ 0\,\mathrm{kA/s} & (8\mathrm{s}, 10\mathrm{s}) \\ 15\,\mathrm{kA/s} & (10\mathrm{s}, 12\mathrm{s}) \end{cases}$$

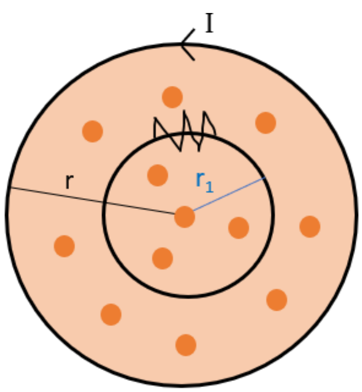
And so E will be:

$$E = 10^{-1}\,\mathrm{N/C}\times\begin{cases} -39 & (0\mathrm{s}, 2\mathrm{s}) \\ 0 & (2\mathrm{s}, 4\mathrm{s}) \\ 63 & (4\mathrm{s}, 8\mathrm{s}) \\ 0 & (8\mathrm{s}, 10\mathrm{s}) \\ -24 & (10\mathrm{s}, 12\mathrm{s}) \end{cases}$$

Filling this into our formula for E, we have:



(b) Say we place a 50 turn wire loop connected to a 10Ω resistor inside the solenoid at the same radius r_1. What is the max current that flows through the resistor during this interval? What is the max power?



So current is I = ξ/R. The emf can be obtained from $d\Phi_B/dt$, or we can just calculate ξ directly from -∫**E**·d**r**.

$$\xi = -\oint \mathbf{E}\cdot d\mathbf{r} = -E\oint dr = -E\cdot(50)(2\pi r_1) = -E\cdot(50)(2\pi\cdot 0.05) = -15.7E$$

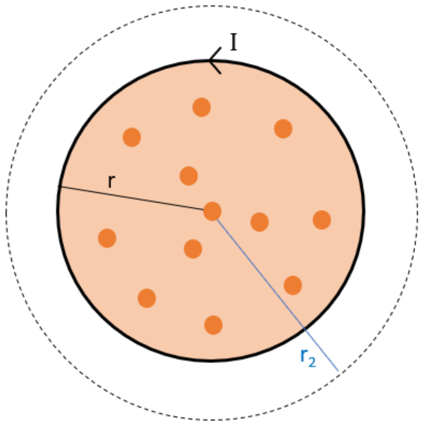
We can ignore the (-) since we already know the direction of the current, as it follows the direction of the field. So,

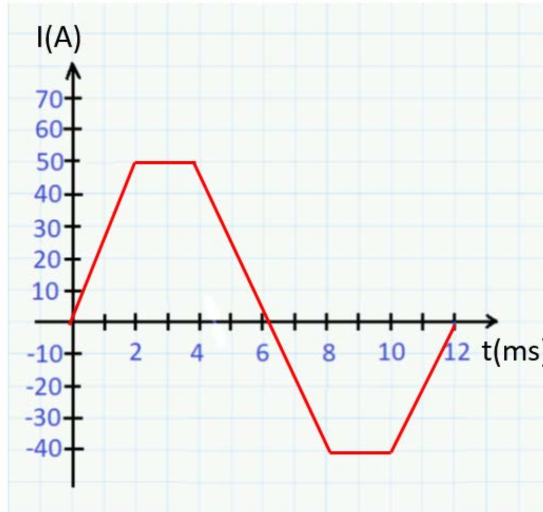
$$I_{\max} = \frac{\xi_{\max}}{R} = \frac{15.7E_{\max}}{10\Omega} = \frac{(15.7)(63\times10^{-1}\,\mathrm{N/C})}{10\Omega} = 9.9\mathrm{A}$$

And the max power would be:

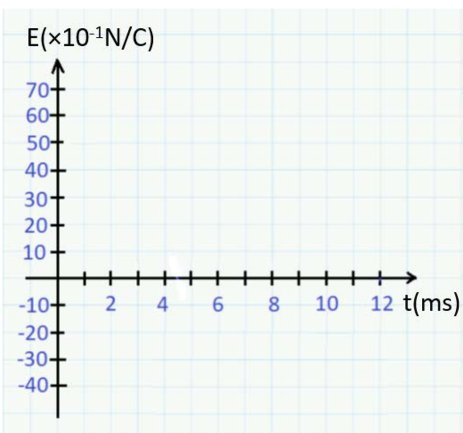
$$P_{\max} = I_{\max}^2 R = (9.9\mathrm{A})^2(10\Omega) = 980\mathrm{W}$$

Problem 2. Consider our same solenoid again. This time let's consider points outside its radius.





(a) Let r_2 = 15cm. Plot the electric field induced along this radius as a function of time (+ corresponds to CCW, and – to CW).



First we need to know what the field running through the solenoid is. And this is still,

$$B = 20\pi \times 10^{-4} I$$

Then from Faraday's law,

$$\oint \mathbf{E}\cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}$$

$$\oint E dr = -\frac{d(\mathbf{B}\cdot\mathbf{A})}{dt}$$

$$E\cdot 2\pi r_2 = -\frac{dB}{dt}A$$

$$E = -\frac{A}{2\pi r_2}\frac{dB}{dt} = -\frac{\pi r^2}{2\pi r_2}\frac{dB}{dt} = -\frac{r^2}{2r_2}\frac{dB}{dt}$$

$$= -\frac{(0.10)^2}{2(0.15)}\cdot 20\pi\times 10^{-4}\frac{dI}{dt} = -\frac{0.05}{2}\cdot 20\pi\times 10^{-4}\frac{dI}{dt} = -2.1\times 10^{-4}\frac{dI}{dt}$$

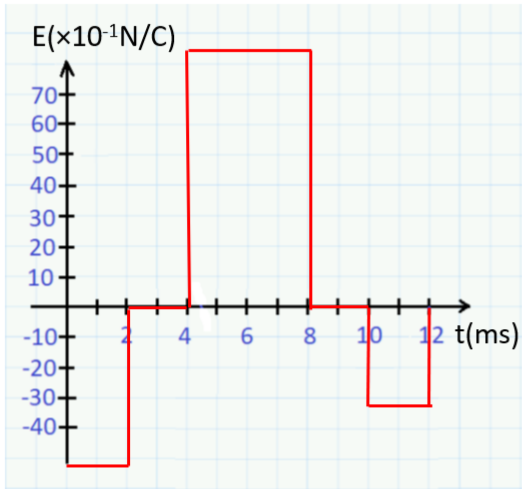
Of course dI/dt is just the slope of the I vs. t graph, which is

$$\frac{dI}{dt} = \begin{cases} 25\,\text{kA/s} & (0\text{s}, 2\text{s}) \\ 0\,\text{kA/s} & (2\text{s}, 4\text{s}) \\ -40\,\text{kA/s} & (4\text{s}, 8\text{s}) \\ 0\,\text{kA/s} & (8\text{s}, 10\text{s}) \\ 15\,\text{kA/s} & (10\text{s}, 12\text{s}) \end{cases}$$

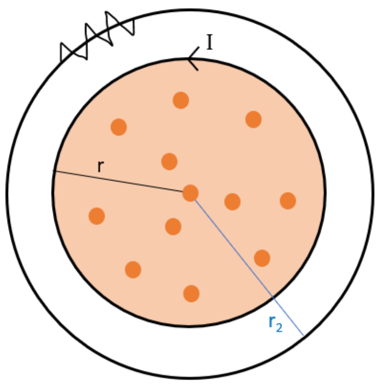
And so E will be:

$$E = 10^{-1}\,\text{N/C}\times\begin{cases} -52 & (0\text{s}, 2\text{s}) \\ 0 & (2\text{s}, 4\text{s}) \\ 84 & (4\text{s}, 8\text{s}) \\ 0 & (8\text{s}, 10\text{s}) \\ -31 & (10\text{s}, 12\text{s}) \end{cases}$$

Filling this into our formula for E, we have:



(b) Say we place a 50 turn wire loop connected to a 10Ω resistor outside the solenoid at the same radius r_2. What is the max current that flows through the resistor during this interval? What is the max power?



So current is I = ξ/R. The emf can be obtained from $d\Phi_B/dt$, or we can just calculate ξ directly from -∫**E**·d**r**.

$$\xi = -\oint \mathbf{E}\cdot d\mathbf{r} = -E\oint dr = -E\cdot(50)(2\pi r_2) = -E\cdot(50)(2\pi\cdot 0.15) = -47E$$

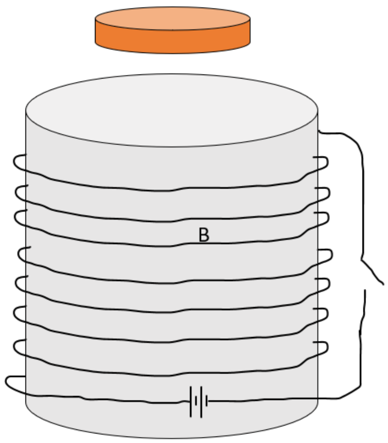
We can ignore the (-) since we already know the direction of the current, as it follows the direction of the field. So,

$$I_{\max} = \frac{\xi_{\max}}{R} = \frac{47E_{\max}}{10\Omega} = \frac{(47)(84\times10^{-1}\,\text{N/C})}{10\Omega} = 39\text{A}$$

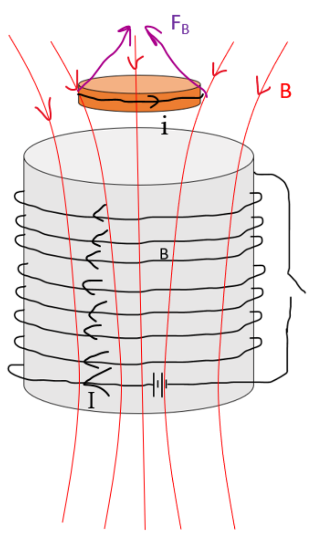
And the max power would be:

$$P_{\max} = I_{\max}^2 R = (39\text{A})^2(10\Omega) = 1520\text{W}$$

Problem 3. Say we hold a copper ring above a solenoid. (a) When we flip the switch a current will be induced in the ring (via Faraday's law). Determine the direction of the resultant force the solenoid will exert on the ring.



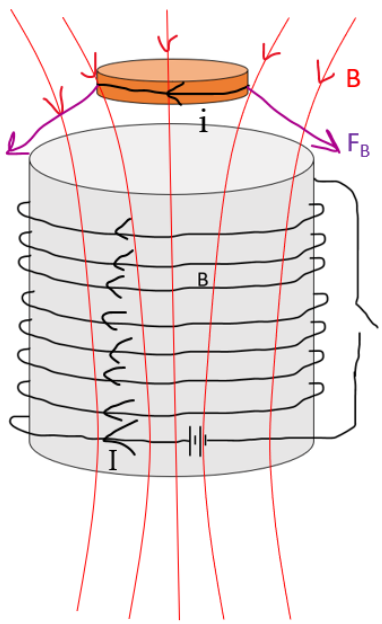
When flip switch, current will flow clockwise around the solenoid, creating a magnetic field flaring inward as shown. The increasing flux in the downward direction will induce an electric field counter-clockwise in the ring (Faraday's law: point thumb in direction of $d\Phi_B/dt$, i.e., downward, then flip hand and fingers curl in direction of induced current). And so current will be induced counter-clockwise too. The solenoid's B field will exert a force on this current $\mathbf{F}_B = \ell\mathbf{I}\times\mathbf{B}$, that points upward/inward. So the ring will be repelled.



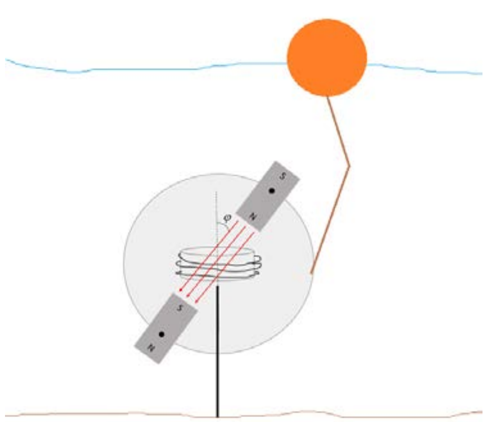
(b) when the switch has been closed for a long time what will be the force?
Once the switch has been closed for a while (a second or so), the magnetic field will be steady. And if its no longer changing, then there will be no changing flux, and so no induced electric field. And so no current, and so no force.

(c) Then if we open the switch what will be the direction of the force?

When open switch, the current in the solenoid will decay. The decreasing downward flux will induce an electric field clockwise in the ring (Faraday's law: point thumb in direction of $d\Phi_B/dt$, i.e., upward, then flip hand and fingers curl in direction of induced current). And so current will be induced clockwise too. The solenoid's B field will exert a force on this current $\mathbf{F}_B = \ell\mathbf{I}\times\mathbf{B}$, that points downward/outward. So the ring will be attracted.



Problem 4. A mechanism for generating power from the sea has been proposed. Basically you attach a a buoy to a wheel on which two magnets are mounted over a bunch of wire loops. As the waves roll in, the buoy will bob up and down, rotating the wheel and magnets, changing the magnetic flux through the loops, and thus generating current. Say we have 200 such wire loops with circumference 15cm, the magnetic field is 2T, and the frequency of oscillation of the buoy is 1Hz. Finally let's say we place 50 000 of these out in the ocean.



(a) What would be the total emf generated by these contraptions as a function of time?
So Faraday's law says,

$$\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}$$
$$-\oint \mathbf{E} \cdot d\mathbf{r} = \frac{d\Phi_B}{dt}$$
$$\xi_{induced} = \frac{d\Phi_B}{dt}$$

So the emf induced in one of the loops is:

$$\begin{aligned}
\xi_{induced\,(loop)} &= \frac{d\mathbf{B} \cdot \mathbf{A}}{dt} \\
&= \frac{dBA\cos(180-\varphi)}{dt} \qquad \text{technically angle between } \mathbf{B} \text{ and } \mathbf{A} \text{ is } 180-\varphi \\
&= -\frac{dBA\cos\varphi}{dt} \\
&= -BA \cdot -\sin\varphi \cdot \frac{d\varphi}{dt} \\
&= BA\sin(\omega t)\omega
\end{aligned}$$

Now we need the area and ω. So $A = \pi r^2$, and $C = 2\pi r$. So $A = \pi(C/2\pi)^2 = C^2/4\pi = (0.15\text{m})^2/4\pi = 1.8\times10^{-3}\text{m}^2$. And $\omega = 2\pi f = 2\pi(1) = 6.3\text{rad/s}$. And of course B = 2T. So,

$$\xi_{induced\,(loop)} = (2\text{T})(1.8\times10^{-3}\text{m}^2)\sin(6.3t)(6.3\text{rad/s}) = 23\text{mV}\sin(6.3t)$$

Then in the entire wire, all 200 loops, we'd have:

$$\xi_{induced\,(all\,loops)} = 200\left[23\text{mV}\sin(6.3t)\right] = 4.5\text{V}\sin(6.3t)$$

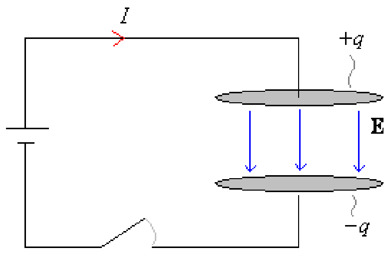
And then multiplying by all contraptions,

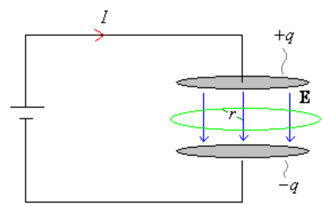
$$\xi_{induced\,(total)} = 50\,000 \cdot \xi_{induced\,(all\,loops)} = 230\text{kV}\sin(6.3t)$$

(b) If we connect our contraption to some huge device which has an effective resistance of 1MΩ, what max power would it deliver?
So Ohm's law:

$$P_{\max} = \frac{\xi_{\max}^2}{R} = \frac{(230\times10^3\text{V})^2}{1\times10^6\Omega} = 53\text{kW}$$

Problem 5. Consider the following circuit, a parallel plate capacitor (radius R) connected to a battery. Suppose that at this instant, current I is flowing through the wire. We know that the B field a distance r away from the long straight wires in the circuit is $B = \mu_0 I/2\pi r$. But what is the induced B field in between the capacitor plates ? Derive a symbolic expression in terms of I and r.

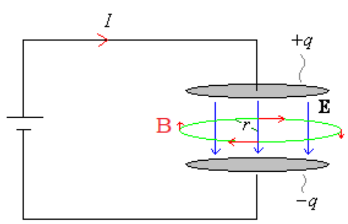




Well, while it is charging, a current I will flow through the circuit, depositing charge q on the plates. This will set up an electric field between as illustrated.

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0}$$

where A is the area of a capacitor plate. Now since q will be changing with time (as the plates charge up), so too will **E**. The changing **E** will induce a magnetic field. What is this **B** in between the plates? Whatever **B** is, it should be cylindrically symmetric. So we draw an Amperean curve (circle) around the **E** field lines, with radius r > R.



Now we'll look for the strength of the induced **B** at the radius r. We use the Maxwell-Ampere law again,

$$\oint \mathbf{B}_{induced} \cdot d\mathbf{r} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint B dr = \mu_0 \varepsilon_0 \frac{d(\mathbf{E} \cdot \mathbf{A})}{dt}$$

$$B \oint dr = \mu_0 \varepsilon_0 \frac{d(EA)}{dt}$$

$$B 2\pi r = \mu_0 \varepsilon_0 A \frac{dE}{dt}$$

$$B = \mu_0 \varepsilon_0 \frac{A}{2\pi r} \frac{dE}{dt}$$

Note that the area, A, appearing in the flux equation is the area of the capacitor plates. This is because **E** doesn't exist outside of the capacitor plates (roughly) so the area of the flux is just the area of the capacitor plates. So we have,

$$B = \frac{dE}{dt}\frac{A\mu_0\varepsilon_0}{2\pi r}$$

Now what is dE/dt? Well,

$$\frac{dE}{dt} = \frac{d\sigma / \varepsilon_0}{dt} = \frac{d(q/A)\varepsilon_0}{dt} = \frac{1}{A\varepsilon_0}\frac{dq}{dt}$$

Now recall that the current in the wire is simply.

$$I = \frac{dq}{dt}$$

and so we have,

$$\frac{dE}{dt} = \frac{I}{A\varepsilon_0}$$

Filling this into our formula for B we have,

$$B = \frac{I}{A\varepsilon_0}\frac{A\mu_0\varepsilon_0}{2\pi r} = \frac{\mu_0 I}{2\pi r}$$

So the field is the same as that for a long straight wire.